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Near-Identity Diffeomorphisms and Exponential ϵ-Tracking and ϵ-Stabilization of First-Order Nonholonomic $SE(2)$ Vehicles

Reza Olfati-Saber
Control and Dynamical Systems
Division of Engineering and Applied Science, 107-81
California Institute of Technology
Pasadena, CA 91125
olfati@cds.caltech.edu

Abstract

In this paper, we address ϵ-tracking and ϵ-stabilization for a class of $SE(2)$ autonomous vehicles with first-order nonholonomic constrains. We introduce a class of transformations called near-identity diffeomorphisms that allow dynamic partial feedback linearization of the translational dynamics of this planar vehicle. This allows us to achieve global exponential ϵ-stabilization and ϵ-tracking (in position) for the aforementioned class of planar vehicles using a coordinate-independent dynamic state feedback. This feedback law is only discontinuous w.r.t. the augmented state. We apply our results to ϵ-stabilization/tracking of a nonholonomic mobile robot.

Keywords: nonholonomic systems, nonlinear control, autonomous vehicles, ϵ-stabilization, ϵ-tracking, mobile robots, dynamic partial feedback linearization, dynamic state feedback

1 Introduction

Control of autonomous vehicles is currently an important field of research. Many vehicles of interest including mobile robots [1, 2, 3], surface vessels [4], VTOL and CTOL aircraft [5, 6], hovercraft, CalTech ducted fan [7, 8], helicopters, aircraft, and underwater vehicles are systems moving in $\mathbb{R}^2$ or $\mathbb{R}^3$. The configuration space of these systems (in the simplest form) is $SE(2)$ or $SE(3)$, respectively (where $SE(n)$ denotes the Special Euclidean group of rigid motions in $\mathbb{R}^n$). All these vehicles are control systems with first-order or second-order nonholonomic constraints.

The main purpose of this paper is to address stabilization and tracking problems for the *dynamic model* of a class of $SE(2)$ vehicles with first-order nonholonomic constraints. A similar treatment for tracking and stabilization of $SE(2)$ vehicles with second-order nonholonomic constraints is presented in [9].

Stabilization of the *kinematic model* of special examples of $SE(2)$ vehicles with first-order nonholonomic constraints including a two-wheeled mobile robot and a rolling disk have been addressed by several researchers in the past. These systems were used as benchmark examples for nonlinear control of nonholonomic systems. Among all methods are motion planning using sinusoids [10], applying time-varying change of coordinates to driftless nonholonomic systems [11], local exponential stabilization of homogeneous systems using periodic inputs [12], ϵ-tracking for a nonholonomic integrator [13], and finally the use of discontinuous change of coordinates [1] and quasi-smooth dynamic state feedback for stabilization of chained form systems [14]. Many of the aforementioned results are local, require the use of small inputs, and suffer from undesired singularities and/or lack of stability in the sense of Lyapunov [10, 12, 1]. In contrast, our results are global, the proposed controllers are coordinate-independent, the stabilization/tracking is rather aggressive (i.e. exponential), the control design is directly performed on the dynamic model, and our approach is readily applicable to $SE(2)$ vehicles with second-order nonholonomic constraints [9] and $SE(3)$ vehicles.

The key tool in our approach is a class of transformations called *near-identity diffeomorphisms (NID)* which allow *dynamic partial feedback linearization* of the translational dynamics of $SE(2)$ vehicles. This in turn results in a dynamic state feedback that achieves global exponential ϵ-stabilization and ϵ-tracking in position for the aforementioned class of $SE(2)$ vehicles. As an example, we apply our results to ϵ-

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stabilization/tracking of a wheeled mobile robot.

The outline of the paper is as follows. In section 2, near-identity diffeomorphisms are introduced and the notions of ϵ-stabilization and ϵ-tracking under dynamic state feedback are defined. In section 3, the dynamics of a planar vehicle with nonholonomic velocity constraint is given. Our main results are presented in section 4. In section 5, the application of these results to ϵ-stabilization/tracking of a mobile robot is provided. Finally, in section 6 concluding remarks are made.

2 Near-Identity Diffeomorphisms

Near-identity diffeomorphisms and their applications were first discussed in [15]. Here, we define what we mean by near-identity diffeomorphisms and provide some examples.

Definition 1. *(near-identity diffeomorphism) Let $\psi(x,\lambda) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ be a smooth function and $z = \psi(x,\lambda)$ be a global diffeomorphism in x for all $\lambda \in \mathbb{R}^p$, i.e. there exists a smooth function $\phi(z,\lambda) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ such that*

$$\phi(\psi(x,\lambda),\lambda) = x, \psi(\phi(z,\lambda),\lambda) = z$$

for all x, z, λ. We say $z = \psi(x,\lambda)$ is a near-identity diffeomorphism *iff for all $x \in \mathbb{R}^n$*

$$\psi(x,\lambda) = x \iff \lambda = 0$$

Remark 1. By uniform continuity of $\psi(x,\lambda)$ at $\lambda = 0$ w.r.t. x that belongs to a compact domain $\mathcal{K}$, it follows that

$$\forall \epsilon > 0, \exists \delta > 0 : \|\lambda\| \leq \delta \implies \|\psi(x,\lambda) - x\| \leq \epsilon$$

for all $x \in \mathcal{K}$. In other words, for $\epsilon \ll 1$, $\psi(x,\lambda)$ is in an ϵ-neighborhood of x-thus the name near-identity.

Two simple examples of a near-identity diffeomorphism are as the following:

i) $\psi(x,\lambda) = x + \lambda\vec{u}, \quad x, \vec{u} \in \mathbb{R}^n, \lambda \in \mathbb{R}, \|\vec{u}\| = 1.$

ii) $\psi(x,\lambda) = x + A\lambda, \quad x \in \mathbb{R}^n, \lambda \in \mathbb{R}^p, p < n, A$ has full column rank.

According to remark 1, in case i) $\delta(\epsilon) = \epsilon$ and in case ii) $\delta(\epsilon) = \epsilon/\sigma_{max}(A)$. In this paper, we use NID's that are similar to the one in i).

Now, consider a nonlinear control system with a (state,input) pair $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$ augmented with another nonlinear system with a (input,state) pair $(x,\lambda) \in \mathbb{R}^n \times \mathbb{R}^p$ as the following

$$\begin{array}{rcl} \dot{x} &=& f(x,u) \\ \dot{\lambda} &=& g(x,\lambda) \end{array} \tag{1}$$

and let $z = \psi(x,\lambda)$ be a near-identity diffeomorphism. In new coordinates, we obtain the following (z,λ)-system

$$\begin{array}{rcl} \dot{z} &=& \tilde{f}(z,\lambda,u) \\ \dot{\lambda} &=& \tilde{g}(z,\lambda) \end{array} \tag{2}$$

with obvious definitions of $\tilde{f}, \tilde{g}$.

Definition 2. *(ϵ-stabilization) For a fix $\epsilon > 0$, let $\lambda_f = \delta(\epsilon)$ satisfy the property in remark 1. Let x_0 be a desired equilibrium point of $\dot{x} = f(x,u)$ with $u = 0$. We say the dynamic state feedback*

$$\begin{array}{rcl} u &=& k(x,\lambda) \\ \dot{\lambda} &=& g(x,\lambda), \ \lambda(0) = \lambda_0 \end{array} \tag{3}$$

achieves globally asymptotic ϵ-stabilization *of x_0 for the x-subsystem in (1) iff for the closed-loop (z,λ)-system*

$$\begin{array}{rcl} \dot{z} &=& \tilde{f}(z,\lambda,\tilde{k}(z,\lambda)) \\ \dot{\lambda} &=& \tilde{g}(z,\lambda) \end{array}$$

with $\tilde{k}(z,\lambda) = k(\phi(z,\lambda),\lambda)$, (x_0,λ_f) is a globally asymptotically stable equilibrium in the sense of Lyapunov.

The notion of ϵ-tracking given a dynamic state feedback is defined in a rather simpliar way (see [15, p. 212]).

Definition 3. *(ϵ-tracking) For a fix $\epsilon > 0$, let $\lambda_f = \delta(\epsilon)$ satisfy the property in remark 1. Consider the following nonlinear system with an (input,output) pair (u,y)*

$$\begin{array}{rcl} \dot{x} &=& f(x,u) \\ y &=& h(x) \end{array} \tag{4}$$

after applying a near-identity diffeomorphism $z = \psi(x,\lambda)$ and augmenting the system in (4) with $\dot{\lambda} = g(x,\lambda)$, the dynamics of the augmented system in new coordinates takes the form

$$\begin{array}{rcl} \dot{z} &=& \tilde{f}(z,\lambda,u) \\ \dot{\lambda} &=& \tilde{g}(z,\lambda) \\ y &=& h(\phi(z,\lambda)) \end{array} \tag{5}$$

Let $u = \tilde{k}(z,\lambda,y_d,\dot{y}_d,\ldots,y_d^{(r)})$ be a control law that achieves asymptotic tracking of a desired trajectory $y_d(\cdot)$ for (5) such that $\lambda \to \lambda_f$ as $t \to \infty$. Then, we say

$$u = k(x,\lambda,y_d,\dot{y}_d,\ldots,y_d^{(r)}) := \tilde{k}(\psi(x,\lambda),\lambda,y_d,\dot{y}_d,\ldots,y_d^{(r)})$$

achieves asymptotic ϵ-tracking of the desired trajectory $y_d(\cdot)$ for (4).

3 Kinematic $SE(2)$ Vehicles

A two-wheeled mobile robot depicted in Fig. 1 is an example of an $SE(2)$ vehicle. A coordinate-independent representation of the dynamics of this mobile robot is in the form

$$\begin{array}{rcl} \dot{x} & = & (Re_1)v_1 \\ \dot{R} & = & R\hat{\omega} \\ \dot{v} & = & \tau \end{array} \tag{6}$$

where $(x, R) \in \mathbb{R}^2 \times SO(2) = SE(2)$, R is a rotation matrix in $\mathbb{R}^2$ satisfying $R^T R = I_2$ with $\det(R) = 1$. Also, $v = (v_1, \omega)^T$ and $\hat{\omega}$ is a skew-symmetric matrix given by

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

By a $SE(2)$ *vehicle*, we mean a robot that its configuration space is $SE(2)$. Consider the following near-identity diffeomorphism

$$z = \psi(x, \lambda) := x + \lambda(Re_1) \tag{7}$$

where $\lambda \in \mathbb{R}$, e_i is the ith standard basis in $\mathbb{R}^n$, and R is the rotation matrix. The dynamics of a kinematic $SE(2)$ vehicle in z-coordinates can be expressed as

$$\begin{array}{rcl} \dot{z} & = & R_\lambda v + (R_\lambda e_1)\dot{\lambda} \\ \dot{R}_\lambda & = & R_\lambda \hat{\omega}(\lambda, \dot{\lambda}) \\ \dot{v} & = & \tau \end{array} \tag{8}$$

where $v = (v_1, \omega)^T$ and $R_\lambda, \hat{\omega}(\lambda, \dot{\lambda})$ are given by

$$R_\lambda = [Re_1 | \lambda Re_2], \; \hat{\omega}(\lambda, \dot{\lambda}) = \begin{bmatrix} 0 & -\lambda\omega \\ \frac{1}{\lambda}\omega & \frac{\dot{\lambda}}{\lambda} \end{bmatrix}$$

where $[C_1|C_2|\ldots|C_m]$ denotes an $n \times m$ matrix with columns $C_1, C_2, \ldots, C_m \in \mathbb{R}^n$. The properties of R_λ in the following lemma is stated for the future use.

Lemma 1. *R_λ satisfies the following properties:*

$$\det(R_\lambda) = \lambda, \; (R_\lambda)^{-1} = R^T_{\lambda^{-1}} \tag{9}$$

The proof of Lemma 1 is very elementary. According to Lemma 1, for $\lambda \neq 0$, R_λ is invertible and for $\lambda \neq 1$, $R_\lambda \notin SO(2)$.

Remark 2. In coordinates, R_λ takes the following form

$$R_\lambda = \begin{bmatrix} \cos(\theta) & -\lambda\sin(\theta) \\ \sin(\theta) & \lambda\cos(\theta) \end{bmatrix}$$

We make use of the following Lemma later.

Lemma 2. *Assume $(\dot{z}, \dot{\lambda}) \to 0$ as $t \to \infty$ and $\lambda(t) \neq 0, \forall t$. Then, $\omega(t) \to 0$ as $t \to \infty$ as well.*

Proof: Based on property i) in Lemma 1 and the fact that $\lambda(t) \neq 0$ for all t, $R_{\lambda(t)}$ is invertible for all t and

$$v = R_\lambda^{-1}(\dot{z} - (Re_1)\dot{\lambda})$$

By assumption, the right hand side of the last equation vanishes and thus $v(t) = (v_1(t), \omega(t)) \to 0$ as $t \to 0$. □

4 Main Results

In this section, we present our main ϵ-stabilization and ϵ-tracking results for the class of kinematic $SE(2)$ vehicles.

Proposition 1. *Consider the kinematic $SE(2)$ vehicle in (8) augmented with*

$$\dot{\lambda} = -c_\lambda(\lambda - \epsilon), \quad \lambda(0) > \epsilon > 0, c_\lambda > 0$$

Then, applying the change of coordinates $(q, p) = (z, \dot{z})$ as the following

$$\begin{array}{rcl} q & = & x + \lambda(Re_1) \\ p & = & R_\lambda v + \dot{\lambda}(R_\lambda e_1) \end{array} \tag{10}$$

transforms the dynamics of the system into

$$\begin{array}{rcl} \dot{q} & = & p \\ \dot{p} & = & R_\lambda \tau + R_\lambda \hat{\omega}(\lambda, \dot{\lambda}) + \dot{\lambda} R_\lambda(\hat{\omega}(\lambda, \dot{\lambda}) - c_\lambda I) \\ \dot{R}_\lambda & = & R_\lambda \hat{\omega}(\lambda, \dot{\lambda}) \\ \dot{\omega} & = & \tau_2 \end{array} \tag{11}$$

In addition, the (q, p)-subsystem in (11) is exact dynamic feedback linearizable as

$$\dot{q} = p, \; \dot{p} = u$$

where $q, p, u \in \mathbb{R}^2$ by applying the following invertible change of control

$$\tau = R_\lambda^{-1} u - \hat{\omega}(\lambda, \dot{\lambda})v - \dot{\lambda}(\hat{\omega}(\lambda, \dot{\lambda}) - c_\lambda I)e_1$$

Proof: By direct calculation. □

Now, we are ready to present our main result on ϵ-stabilization of kinematic $SE(2)$ vehicles.

Proposition 2. *(ϵ-stabilization) Any desired position $x_0 \in \mathbb{R}^2$ for the kinematic $SE(2)$ vehicle in (6) can be rendered globally exponentially ϵ-stable by applying the following quasi-smooth dynamic state feedback*

$$\begin{array}{rcl} \tau & = & -c_p R_\lambda^{-1}(q - x_0) - c_d v - c_d \dot{\lambda} e_1 \\ & - & \hat{\omega}(\lambda, \dot{\lambda})v - (\hat{\omega}(\lambda, \dot{\lambda}) - c_\lambda I)\dot{\lambda} e_1 \\ \dot{\lambda} & = & -c_\lambda(\lambda - \epsilon), \quad \lambda(0) > \epsilon \end{array} \tag{12}$$

where $c_p, c_d, c_\lambda > 0$ are constants and q is defined in (10). In addition, $\omega(t) \to 0$ as $t \to \infty$.

Proof: Fix an $\epsilon > 0$, clearly $\lambda = \epsilon$ is globally exponentially stable for the λ-subsystem and $\lambda(t) \geq \epsilon > 0, \forall t \geq 0$. Let $q_0 = x_0$. Based on proposition 1, $\ddot{q} = u$. Thus, applying the following feedback

$$u = -c_p(q - q_0) - c_d p, \quad c_p, c_d > 0$$

guarantees global exponential stability of $(q, p) = (x_d, 0)$. Substituting this feedback law in (1) gives the dynamic state feedback in the question. Since both $p = \dot{z}$ and $\dot{\lambda}$ are exponentially vanishing, from Lemma 2, it follows that $\omega(t) \to 0$ as $t \to \infty$. □

Remark 3. Based on dynamics of the kinematic $SE(2)$ vehicle in (6), if $\tau_1 = v_1 = 0$, x remains invariant and in local coordinates $\ddot{\theta} = \tau_2$. Therefore, the orientation of the vehicle can be exponentially stabilized to any desired orientation without changing the position. Hence, it is sufficient to only stabilize the translational dynamics of a kinematic $SE(2)$ vehicle as in Proposition 2 and then switch to a local controller that changes the attitude of the vehicle to a desired attitude.

Exponential ϵ-tracking of a smooth desired output $x_d(\cdot)$ can be obtained based on the following result.

Proposition 3. *(ϵ-tracking) Let $x_d(t) : \mathbb{R} \to \mathbb{R}^2$ be a C^2 smooth desired trajectory. Then, the following dynamic state feedback law achieves global exponential ϵ-tracking for the desired output (i.e. position) $x_d(\cdot)$ of the kinematic $SE(2)$ vehicle in (6)*

$$\begin{array}{rcl} \tau & = & -c_p R_\lambda^{-1} z - c_d v - c_d \dot{\lambda} e_1 \\ & - & \hat{\omega}(\lambda, \dot{\lambda}) v - (\hat{\omega}(\lambda, \dot{\lambda}) - c_\lambda I)\dot{\lambda} e_1 \\ & + & R_\lambda^{-1}(c_p x_d + c_d \dot{x}_d + \ddot{x}_d) \\ \dot{\lambda} & = & -c_\lambda(\lambda - \epsilon) \end{array} \tag{13}$$

where $c_p, c_d, c_\lambda > 0$ are constants and q is defined in (10).

Proof: Let us consider the following partial state-output feedback law

$$u = -c_p(q - x_d) - c_d(p - \dot{x}_d) + \ddot{x}_d, \quad c_p, c_d > 0$$

and define the output error as $e = q - x_d$. Then, e satisfies the following output error dynamics

$$\ddot{e} + c_d \dot{e} + c_p = 0$$

Since $c_p, c_d > 0$, the output error e globally exponentially converges to zero. This means that $x(t)$ globally exponentially converges to a λ-neighborhood of $x_d(t)$ where λ exponentially converges to ϵ. □

5 Example: A Mobile Robot

Consider the mobile robot depicted in Fig. 1 [1, 2]. The robot has two rolling wheels that can be controlled independently using input torques. The dynamics of this nonholonomic mobile robot in coordinates is given in equation (6).

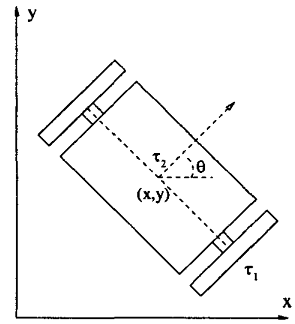

Figure 1: A mobile robot

Figures 2 and 3 show the path of the mobile robot starting at position $x = (2, 3)^T$ for the initial orientation angles $\theta = k\pi/4, k = 0, \ldots, 7$. The trajectories for the position and input torques (controls) are shown in Figure 4. These results demonstrate that the controller aggressively stabilizes the origin for this nonholonomic mobile robot. Each trajectory exponentially converges to a point within a distance $\epsilon = 0.01$ from the desired equilibrium $x_d = 0$. This is sufficiently close to the origin for all practical purposes. The values of the parameters in all simulations for the mobile robot were chosen as $c_\lambda = 1, c_p = 1, c_d = 2, \lambda(0) = 0.5, \epsilon = 0.01$. The trace trajectories of this nonholonomic robot are shown in Fig. 5. Also, Fig. 6 demonstrates simulation results of the ϵ-tracking for a nonholonomic robot starting at position $(4, 4)$ with orientation angle $\pi/2$. The desired trajectory is an ellipse $(x, y) = (3\sin t, 4\cos t)$. Clearly, the robot very quickly converges to an ϵ-neighborhood of the desired trajectory.

6 Conclusion

In this paper, we introduced a class of diffeomorphisms referred to as near-identity diffeomorphisms which allow dynamic partial feedback linearization of the translational dynamics of certain classes of $SE(2)$ vehicles. This in turn led to global exponential ϵ-stabilization and ϵ-tracking in position for these planar vehicles. The main features of the obtained control laws are that they are coordinate independent (i.e. require no switching of the local charts), and are directly designed for the dynamic model of a vehicle as compared to the kinematic model. We applied our results to tracking and stabilization of a wheeled nonholonomic mobile robot. The simulation results demonstrate that the dynamic state feedback used for ϵ-stabilization and ϵ-tracking of this mobile robot is an aggressive control law (i.e. the solution converges exponential fast to the desired position/trajectory).

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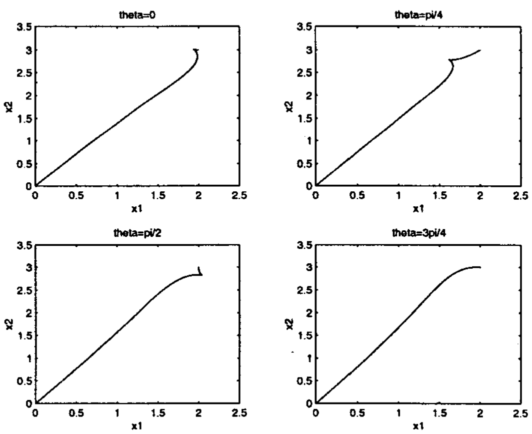


Figure 2: Trajectories of the nonholonomic mobile robot in (x_1, x_2)-plane for initial position $x = (2,3)^T$ $(v = 0)$ and orientation angles $\theta = 0, \pi/4, \pi/2, 3\pi/4$.

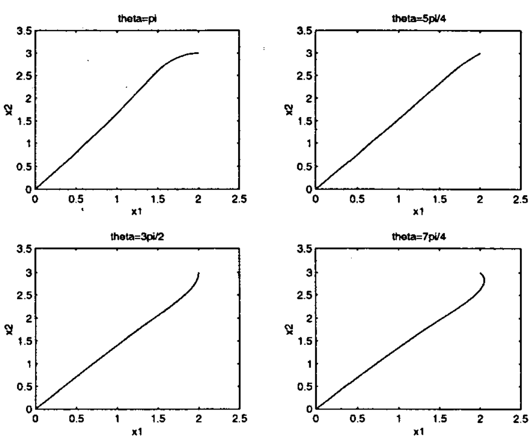


Figure 3: Trajectories of the nonholonomic mobile robot in (x_1, x_2)-plane for initial position $x = (2,3)^T$ $(v = 0)$ and orientation angles $\theta = \pi, 5\pi/4, 3\pi/2, 7\pi/4$.

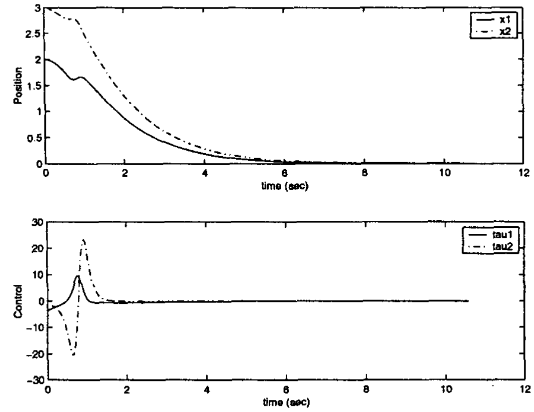
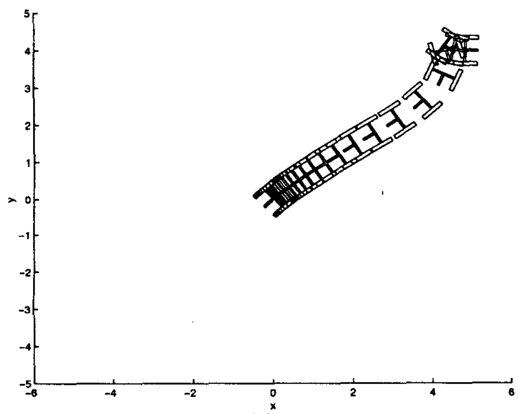
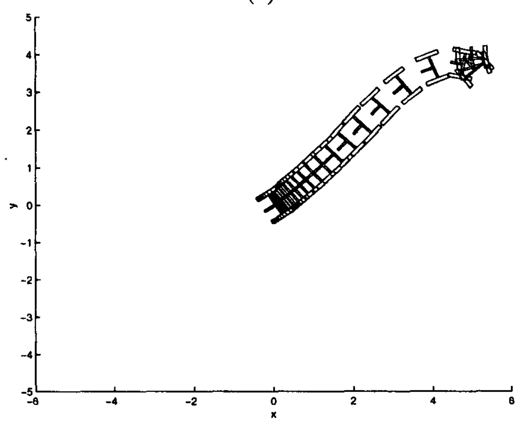


Figure 4: Trajectories of the nonholonomic mobile robot in (x_1, x_2)-plane starting at $x = (2, 3)^T$ $(v = 0)$ with angle $\theta = \pi/4$

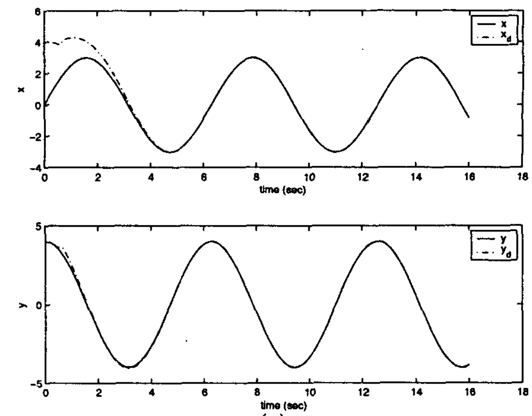


(a)

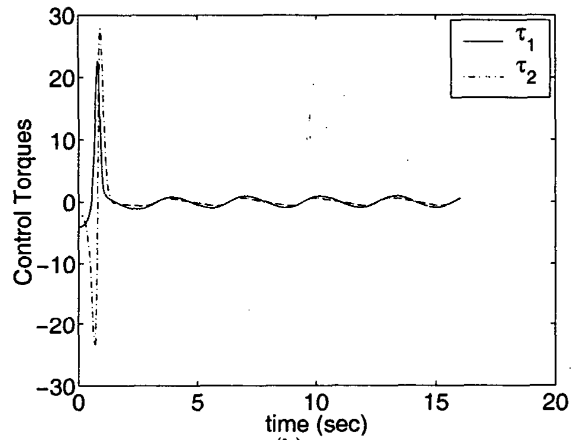


(b)

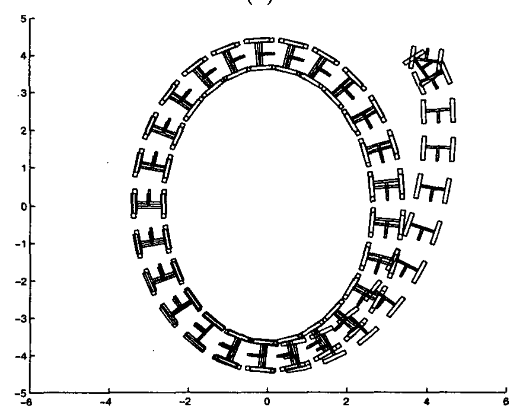
Figure 5: Trace trajectories of a two-wheel nonholonomic robot with initial position $(5, 4)$ and heading angles (a) $\theta = 0$ and (b) $\theta = \pi/2$.



(a)



(b)



(c)

Figure 6: (a),(b) Position trajectory and control for tracking starting at the initial point $(4, 4, \pi/2)$ and (c) The associated trace trajectories of the nonholonomic robot for initial condition in (a).